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Construction of a function using its values along C^1 curves

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Abstract. Let $G : D \subset R^n \to R$ be a function. Any parametrized curve α in D determines the composition $g_\alpha = G \circ \alpha$. If α belongs to a family of curves, the family $\{g_\alpha\}$ satisfies some conditions. Our goal is to find the conditions in which the families $\{\alpha\}$, $\{g_\alpha\}$ determine the function G.

Section 1 emphasizes the origin of the problem. Section 2 defines and studies the notion of the Γ-function. Section 3 presents the construction of a function using a Γ-function.

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1 The origin of the problem

In the theory of nonholonomic optimization [6] it appears the following types of problems. Let D be an open set of R^n and $\omega = \sum_{i=1}^n \omega_i dx^i$ be a C^0 Pfaff form on D. For every parametrized C^1 curve $\alpha : I \to D$, we consider $g_\alpha : I \to R$, $g_\alpha(t) = \int_{t_0}^t \langle \omega(\alpha(u)), \alpha'(u)\rangle du + c_\alpha$ (a primitive of ω along α). In this way we obtain a family of functions $\{g_\alpha\}$, called *system of* ω*-primitives* which depends on the family of constants $\{c_\alpha\}$. Question: is it possible to choose the family $\{c_\alpha\}$ such that $g_{\alpha\circ\varphi} = g_\alpha \circ \varphi$ for any α and for any diffeomorphism φ?.

If $\omega = dG$, with $G : D \to R$ a C^1 function, the answer is positive, because we can consider $g_\alpha = G \circ \alpha$. In this way, it appears a more general problem. Let us suppose that for any parametrized curve $\alpha : I \to D$, a function $g_\alpha : I \to R$ is given. What conditions we must impose to the family $\{g_\alpha\}$ in order to exist a unique function $G : D \to R$, having certain properties (like continuity, with partial derivatives, class C^1) and such that $G \circ \alpha = g_\alpha$?

Recall that two C^k parametrized curves $\alpha : I \to D$ and $\beta : J \to D$ are said to be *equivalent*, if there exists a C^k diffeomorphism $\varphi : J \to I$ such that $\beta = \alpha \circ \varphi$. We say that φ is a *change of parameter* on α. An equivalence class $\tilde{\alpha}$ of a given C^k parametrized curve α is called *curve*. Then α is called a representative of $\tilde{\alpha}$.

Let $I = [a, b]$ be a closed interval in R. A continuous mapping $\alpha : I \to D$ is said to be a *piecewise C^1 parametrized curve* if there exists a division $a = t_0 < t_1 < \cdots < t_p = b$ of the interval I so that restriction of α to each subinterval $[t_i, t_{i+1}]$, $i = \overline{0, p-1}$ is a C^1 function. If I is an arbitrary interval, the previous definition is extended in an obvious way.

2 Γ-functions

We denote by $\Gamma^0(D)$ the family of all the C^0 parametrized curves in D and by $\Gamma^1(D)$ the family of all the piecewise C^1 parametrized curves in D. Let $G : D \subset R^n \to R$ be a C^1 function. For each $\alpha \in \Gamma^1(D)$, we consider the function $g_\alpha = G \circ \alpha$, which is an element of $\Gamma^1(R)$. In this way we produce a family $\{g_\alpha\}$ which has properties of the following type:

(a) For any $\alpha \in \Gamma^1(D)$, the functions α, g_α have the same domain of definition. Also, for a parametrized piecewise C^1 curve α, the following statements are true: (1) the function g_α is a piecewise C^1 function; (2) if α is a C^1 function in a neighborhood of a point t_0, then g_α is a C^1 function in the same neighborhood.

(b) If α and $\beta = \alpha \circ \varphi$ are equivalent parametrized curves, then $g_\beta = g_\alpha \circ \varphi$.

(c) If $\alpha \in \Gamma^1(D)$, $\alpha : I \to D$, and J is a subinterval of I, then $g_{\alpha|J} = g_\alpha|J$.

(d) For any $x \in R^n$ and each $i = \overline{1, n}$, we define the parametrized axis $\alpha_x^i(t) = x + te^i$, $\forall t \in (-\varepsilon_i, \varepsilon_i)$, where $e^i = (0, \ldots, 1, \ldots 0)$. Obviously, $g'_{\alpha_x^i}(0) = \frac{\partial G}{\partial x^i}(x)$. In this way, it follows that the function $h^i : D \to R$ by $h^i(x) = g'_{\alpha_x^i}(0)$ is continuous.

In the Section 3, we shall show that previous properties are sufficient to recover the function G from the family $\{g_\alpha\}$.

Let us consider $g : \Gamma^1(D) \to \Gamma^k(R)$, $k = \overline{0, 1}$ an arbitrary mapping. For each $\alpha \in \Gamma^1(D)$ we denote by g_α the element $g(\alpha) \in \Gamma^k(R)$. For this kind of functions we can consider some axioms.

(A_0) If $\alpha \in \Gamma^1(D)$, then $\mathrm{dom}(\alpha) = \mathrm{dom}(g_\alpha)$. In addition, if $k = 1$ and if α is a C^1 function in a neighborhood of a point $t_0 \in \mathrm{dom}(\alpha)$, then g_α is also a C^1 function in the same neighborhood.

(A_1) The axiom (A_0) is satisfied. Moreover, if $\alpha \in \Gamma^1(D)$ and φ is a change of parameter on α, then $g_{\alpha \circ \varphi} = g_\alpha \circ \varphi$.

(A_2) The axiom (A_0) is satisfied. Moreover, if $\alpha \in \Gamma^1(D)$ with $\mathrm{dom}(\alpha) = I$, then $g_{\alpha|J} = g_\alpha|J$ for every subinterval J in I.

In the case $k = 1$ we can consider one more axiom, as follows. Let $g : \Gamma^1(D) \to \Gamma^1(R)$ be a function which fulfils (A_2). Then, for each $i = \overline{1,n}$ we consider $h^i : D \to R$ by $h^i(x) = g'_{\alpha_x^i}(0)$, where $\alpha_x^i(t) = x + te^i$, $\forall t \in (-\varepsilon_i, \varepsilon_i)$, and $e^i = (0, \dots, 1, \dots 0)$. Taking into account the axiom (A_2), it results that the function $g_{\alpha_x^i}$ does not depend on ε_i in a neighborhood of 0, so the number $h^i(x)$ is well defined.

(A_3) The axiom (A_2) is satisfied and, in addition, for every $i = \overline{1,n}$ and for every $\alpha \in \Gamma^1(D)$, it results that $h^i \circ \alpha \in \Gamma^0(R)$.

1 Example. For each $\alpha \in \Gamma^1(D)$ we choose $x_0 \in \operatorname{Im}\alpha$ and $t_0 \in \operatorname{dom}(\alpha)$ such as $\alpha(t_0) = x_0$. We can easily see that the mapping $g : \Gamma^1(D) \to \Gamma^1(R)$ defined by $g_\alpha(t) = \int_{t_0}^{t} ||\alpha'(u)||du$ satisfies (A_0), but does not satisfy (A_1) and (A_2).

2 Example. Let $G : D \to R$ be a C^k function, where $k = \overline{0,1}$. Now we consider $g : \Gamma^1(D) \to \Gamma^k(R)$ defined by $g_\alpha = G \circ \alpha$. It is obvious that g fulfills (A_1) and (A_2).

3 Example. Let us consider $g : \Gamma^1(D) \to \Gamma^1(R)$ defined by $g_\alpha(t) = t$, $t \in \operatorname{dom}(\alpha)$. Obviously, g satisfies (A_2) and (A_3), but does not satisfy (A_1).

4 Example. Let us consider $g : \Gamma^1(D) \to \Gamma^k(R)$, $k = \overline{0,1}$ a function defined as follows $g_\alpha(t) = 0$ $\forall t \in \operatorname{dom}(\alpha)$, if $\operatorname{Im}\alpha$ is included in straight line and $g_\alpha(t) = 1$, $\forall t \in \operatorname{dom}(\alpha)$, otherwise. Obviously, g satisfies (A_1), but does not satisfy (A_2).

From the previous examples it follows that (A_1) and (A_2) are independent axioms and no one is equivalent to (A_0). Also, in the example of Section 3, we shall prove that (A_3) is independent with respect to (A_1) and (A_2).

5 Definition. A mapping $g : \Gamma^1(D) \to \Gamma^k(R)$, $k = \overline{0,1}$ which satisfies the axiom (A_0) is called Γ-*function*.

6 Remark. Let $g : \Gamma^1(D) \to \Gamma^k(R)$, $k = \overline{0,1}$ be a Γ- function which satisfies the axiom (A_1). If α and $\beta = \alpha \circ \varphi$ are two equivalent parametrized curves of $\Gamma^1(D)$ and $t_0 = \varphi(u_0)$, then $g_\alpha(t_0) = g_\beta(u_0)$.

7 Proposition. *Let $g : \Gamma^1(D) \to \Gamma^k(R)$, $k = \overline{0,1}$ be a Γ-function which satisfies the axiom (A_1) and (A_2). Let $\alpha_1 : I_1 \to D$ and $\alpha_2 : I_2 \to D$ be two parametrized curves of $\Gamma^1(D)$ such there exist $t_1 \in I_1$ and $t_2 \in I_2$ with $\alpha_1(t_1) = \alpha_2(t_2)$. Then $g_{\alpha_1}(t_1) = g_{\alpha_2}(t_2)$.*

Proof. Let us consider $\beta_1 = \alpha_1 \circ \varphi_1 : J_1 \to D$ and $\beta_2 = \alpha_2 \circ \varphi_2 : J_2 \to D$ two parametrized curves of $\Gamma^1(D)$ which are equivalent to α_1, α_2 respectively, such as there exist the real numbers $a < b < c$ satisfying the following conditions: $K_1 = [a, b] \subset J_1$, $K_2 = [b, c] \subset J_2$, $\varphi_1(b) = t_1$ and $\varphi_2(b) = t_2$. By the previous remark it follows $g_{\alpha_1}(t_1) = g_{\beta_1}(b)$ and $g_{\alpha_2}(t_2) = g_{\beta_2}(b)$. Consider now $\gamma : K_1 \cup K_2 \to D$ defined by $\gamma|K_1 = \beta_1|K_1$ and $\gamma|K_2 = \beta_2|K_2$. By using the axiom (A_2), we obtain: $g_\gamma|K_1 = g_{\gamma|K_1} = g_{\beta_1|K_1} = g_{\beta_1}|K_1$ and $g_\gamma|K_2 = g_{\gamma|K_2} = g_{\beta_2|K_2} = g_{\beta_2}|K_2$. Consequently, we have $g_\gamma(b) = g_{\beta_1}(b) = g_{\beta_2}(b)$, i.e., $g_{\alpha_1}(t_1) = g_{\alpha_2}(t_2)$. $\boxed{QED}$

8 Corollary. *Let $g : \Gamma^1(D) \to \Gamma^k(R)$, $k = \overline{0,1}$, be a Γ-function which satisfies the axioms (A_1) and (A_2). Then for any $\alpha \in \Gamma^1(D)$ and $t_1, t_2 \in \mathrm{dom}(\alpha)$ with $\alpha(t_1) = \alpha(t_2)$ we have $g_\alpha(t_1) = g_\alpha(t_2)$.*

3 Construction of a function using a Γ-function

In what follows we shall use the next result ([1], [2], [5]):

9 Theorem. *Let (x_n) be a sequence of distinct points of R^p which converges to the limit $a \in R^p$. Then, there exist a subsequence (x_{n_k}), a simple C^1 parametrized curve α, regular at the point a, and a sequence of real numbers $t_k \to 0$ such that $\alpha(t_k) = x_{n_k}$ and $\alpha(t_0) = a$.*

10 Lemma. *Let $G : D \to R$ be a function.*

(a) Let us suppose that for every simple parametrized curve α of $\Gamma^1(D)$ the function $G \circ \alpha$ is continuous. Then, G is a continuous function.

(b) Let us suppose that for every simple C^1 parametrized curve $\alpha \in \Gamma^1(D)$ the function $G \circ \alpha$ is a C^1 function. Then G is a continuous function that has first order partial derivatives.

Proof. (a) Let (x_n) be a sequence of D such that $x_n \to a \in D$. By absurdum, we suppose that $G(x_n) \nrightarrow G(a)$, i.e., there exists a subsequence (y_n) of (x_n) such as $G(y_n) \to l$ with $l \neq G(a)$. Applying the previous Theorem we obtain a subsequence (z_n) of (y_n), a simple parametrized curve $\alpha \in \Gamma^1(D)$ and a sequence (t_n) of R such that, $z_n \to a$, $\alpha(t_n) = z_n$, $\alpha(0) = a$ and $t_n \to 0$. Due to continuity of the function $G \circ \alpha$ we obtain the contradiction $G(z_n) \to G(a)$.

(b) Taking as α the natural parametrizations of each coordinate axis, it follows that G has first order partial derivatives. $\boxed{QED}$

11 Theorem. *(1) Let us assume that the Γ-function $g : \Gamma^1(D) \to \Gamma^0(R)$ satisfies the axioms (A_1) and (A_2). Then, there exists a unique continuous function $G : D \to R$ such that for every $\alpha \in \Gamma^1(D)$ we have $G \circ \alpha = g_\alpha$.*

(2) Conversely, for any continuous function $G : D \to R$ there exists a unique Γ-function $g : \Gamma^1(D) \to \Gamma^0(R)$ which satisfies the axioms (A_1) and (A_2) and such that $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma^1(D)$.

Proof. Let $g : \Gamma^1(D) \to \Gamma^0(R)$ be a Γ-function which fulfills the axioms (A_1) and (A_2). We define a function $G : D \to R$ as follows: if $x \in D$ and

$\alpha \in \Gamma^1(D)$ with $\alpha(t) = x$, then $G(x) = g_\alpha(t)$. By using the Proposition 7 and the Corollary 8, it follows that G is well defined and unique. It is clear that $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma^1(D)$. Applying the statement (a) from previous Lemma, it follows that G is a continuous function. The converse is obvious. $\boxed{QED}$

12 Remark. The proof works also in case that the functions g_α are not continuous. Obviously, in this case, the function G does not result as a continuous function. Hence, the conditions (A_1) and (A_2) with g_α arbitrary functions, are necessary and sufficient conditions for the existence and uniqueness of a function $G : D \to R$ with $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma^1(D)$.

13 Theorem. *(1) Let $g : \Gamma^1(D) \to \Gamma^1(R)$ be a Γ-function which satisfies the axioms (A_1) and (A_2). Then, there exits a unique continuous function $G : D \to R$, having first order partial derivatives such that $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma^1(D)$.*

(2) Let $G : D \to R$ be a function such that for any simple C^1 parametrized curve $\alpha \in \Gamma^1(D)$ it results that $G \circ \alpha$ is a C^1 function. Then, there exists a unique Γ-function $g : \Gamma^1(D) \to \Gamma^1(R)$ which satisfies the axioms (A_1) and (A_2) and such that $g_\alpha = G \circ \alpha$ for any $\alpha \in \Gamma^1(D)$.

The proof is similar with the previous, excepting that we use the statement (b) in Lemma 10.

14 Theorem. *(1) Let $g : \Gamma^1(D) \to \Gamma^1(R)$ a Γ-function which satisfies the axioms (A_1) and (A_3) (hence also (A_2)). Then, there exists a unique C^1 function $G : D \to R$ such that $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma^1$.*

(2) Conversely, for any C^1 function $G : D \to R$ there exists a unique Γ-function $g : \Gamma^1(D) \to \Gamma^1(R)$ which satisfies the axioms (A_1) and (A_3) and such that $g_\alpha = G \circ \alpha$ for any $\alpha \in \Gamma^1(D)$.

Proof. Let $g : \Gamma^1(D) \to \Gamma^1(R)$ be a mapping which satisfies the axioms (A_1) and (A_3). From the previous Theorem it follows the existence of a continuous function $G : D \to R$, having first order partial derivatives such that $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma^1(D)$. It follows that $\frac{\partial G}{\partial x^i} = h^i$, $i = \overline{1,n}$, where h^i are the functions defined in the axiom (A_3). By using this axiom, it results that $h^i \circ \alpha \in \Gamma^0(R)$ for any $\alpha \in \Gamma^1(D)$. From the statement (a) in Lemma 10 we obtain that h^i is a continuous function for any $i = \overline{1,n}$, namely G is a C^1 function. The converse is obvious. $\boxed{QED}$

15 Example. We shall show that there exists a Γ-function $g : \Gamma^1(D) \to \Gamma^1(R)$ which satisfies the axioms (A_1) and (A_2) but does not satisfy (A_3).

For that, we consider the function $G : R^2 \to R$,

$$G(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & \text{for} \quad (x,y) \neq (0,0) \\ 0, & \text{for} \quad (x,y) = (0,0) \end{cases}$$

Let us show that G fulfills the conditions in Theorem 13. To this aim, we consider a simple parametrized curve $\alpha \in \Gamma^1(D)$ such that $\alpha(0) = (0,0)$. We must prove that $G \circ \alpha$ is a C^1 function. Since α is a simple curve it follows that

$$(G \circ \alpha)'(t) = \frac{x^2(x^2 - y^2)y' + 2xy^3x'}{(x^2 + y^2)^2}(t)$$

for any $t \neq 0$.

First, let us assume that $x'(0) \neq 0$. Since

$$(G \circ \alpha)'(t) = \frac{(1 - (y/x)^2)y' + 2(y/x)^3x'}{[1 + (y/x)^2]^2}(t)$$

for $t \neq 0$, we can apply L'Hospital rule for $\frac{y(t)}{x(t)}$, obtaining the existence and finiteness of the $\lim_{t \to 0} G(\alpha(t))'$. Assume now that $x'(0) = y'(0) = 0$. Since

$$|\,(G \circ \alpha)'\,(t)\,| \leq \left(|y'| + 2|x'|\right)(t),$$

for $t \neq 0$, it t follows that $\lim_{t \to 0}(G \circ \alpha)'(t) = 0$. Finally, we can easily see that the first order partial derivatives of G are not continuous. Thus, by the Theorem 13, the Γ-function $g : \Gamma^1(D) \to \Gamma^1(R)$, $g_\alpha = G \circ \alpha$, will satisfy the axioms (A_1) and (A_2). But g does not satisfy (A_3). Indeed, if g satisfied (A_3), then the Theorem 14 would show that G is a C^1 function, which is a contradiction.

Let $\Gamma_s^1(D)$ the family of all the simple parametrized curves $\alpha \in \Gamma^1(D)$. It is obvious that the Theorems 11, 13 and 14 are also true in the case when we replaced $\Gamma^1(D)$ by $\Gamma_s^1(D)$.

Let $\omega = \sum_{i=1}^n \omega_i(x)dx^i$ be a C^0 Pfaff from on D. For each curve $\tilde{\alpha}$ with $\alpha \in \Gamma_s^1(D)$ we choose a point $x_0 \in \operatorname{Im}\tilde{\alpha}$ and for each $\beta \in \tilde{\alpha}$, $\beta(t_0) = x_0$, we consider $g_\beta(t) = \int_{t_0}^t \langle \omega(\beta(u)), \beta'(u) \rangle du$. In this way, we obtain a Γ-function $g : \Gamma_s^1(D) \to \Gamma^1(R)$ which satisfies the axiom (A_1).

16 Theorem. *The continuous Pfaff form ω is exact if and only if the Γ-function g defined above fulfils the axiom (A_2).*

Proof. Let us suppose that g fulfils (A_2). Applying the Theorem 13 it follows that there exists a continuous function $G : D \to R$ having the first order partial derivatives such that $G \circ \alpha = g_\alpha$ for any $\alpha \in \Gamma_s^1(D)$. It results $\frac{\partial G}{\partial x^i} = \omega^i$, $i = \overline{1,n}$; thus G is a C^1 function and $dG = \omega$. The converse is obvious. $\boxed{QED}$

17 Corollary. *The Γ-function g defined above satisfies (A_2) if and only if g satisfies (A_3).*

Final remark

We consider now the following sets:

$$\begin{aligned}
\mathcal{G}^{\circ}(D) &= \{\, g : \Gamma^1(D) \to \Gamma^0(R) | g \text{ satisfies } (A_1) \text{ and } (A_2) \,\},\\
\mathcal{G}^{1/2}(D) &= \{\, g : \Gamma^1(D) \to \Gamma^1(R) | g \text{ satisfies } (A_1) \text{ and } (A_2) \,\},\\
\mathcal{G}^{1}(D) &= \{\, g : \Gamma^1(D) \to \Gamma^1(R) | g \text{ satisfies } (A_1) \text{ and } (A_3) \,\},\\
C^{1/2}(D) &= \{\, G : D \to R |\, G \circ \alpha \text{ is a } C^1 \text{ function for any simple}\\
&\qquad\qquad \text{parametrized curve } \alpha \in \Gamma^1(D) \,\}.
\end{aligned}$$

Obviously, all these sets are real vector spaces. From the statement (b) in Lemma 10 it follows that $C^{1/2}(D) \subset C^0(D)$.

To each continuous function $G : D \to R$ we can attache the Γ-function g defined by $g_\alpha = G \circ \alpha$, $\forall \alpha \in \Gamma^1(D)$. In this way the Theorems 11, 13 and 14 can be reformulated as

18 Theorem. *The correspondence $G \to g$ above induces the following vector space isomorphisms: $C^0(D) \approx \mathcal{G}^0(D)$, $C^{1/2}(D) \approx \mathcal{G}^{1/2}(D)$ and $C^1(D) \approx \mathcal{G}^1(D)$.*

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